The Paradox of Infinite Limits: A Realist Response

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Abstract

In this paper, we distinguish between different uses of mathematical limits in physics, and we determine the conditions under which an infinite limit should be understood as giving rise to an "infinite idealization", intended as a misrepresentation of the target system by way of introducing an infinite system. We point out that when infinite limits are used as infinite idealizations they can lead one to the Paradox of Infinite Limits, which allegedly poses a threat to scientific realism. In particular, this depends on whether the idealization is *essential* for the explanation of the physical phenomenon under investigation. Instead, other uses of infinite limits such as approximations and abstractions do not raise any substantial problem for scientific realism. We also argue that, even in the case of "essential idealizations", there are ways of coping with the alleged incompatibility between infinite idealizations and scientific realism, which ultimately rely on empirical considerations.

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1 Introduction

Scientific realism is a central topic in philosophy of science. Although there are many different formulations of this concept in the literature, most scientific realists are committed to the idea that we have good reason to believe that the content of our best scientific theories, regarding both observable and unobservable aspects of the world, is true or at least approximately true. According to most realists, scientific realism involves a semantic dimension, according to which one is committed to a literal interpretation of scientific claims about the world (Chakravartty 2017). Perhaps the strongest argument in favor of scientific realism is the "no miracles argument", which asserts that the success of our well established scientific theories would be a miracle if the content of such theories were not true or at least approximately true (Putnam 1975; Boyd 1983). The notion of "approximate truth" plays an important role in current approaches to scientific realism, since it is widely held, even by realists, that our best scientific theories are likely false, strictly speaking. Important technical work has been developed to make the notion of "approximately truth" precise and we will address part of this work in this paper.

As plausible as it is, a scientific realist position faces various difficulties that have casted doubt on the "no miracles argument" and have motivated an anti-realist attitude towards our most successful theories (van Fraassen 1980; Rosen 1994). One of the most outstanding challenges for scientific realism is the use of idealizations in scientific theories, which involves the assumption of fictional systems that are intended to resemble the real-world systems we are interested in (Godfrey-Smith 2009). An important example of idealizations in physics is the use of "infinite idealizations", which involve the introduction of infinite systems that can be constructed by means of mathematical limits that are invoked to explain the behavior of target systems, notwithstanding the fact that the latter are considered to be finite according to our most successful background theories.[1]

The problem with infinite idealizations, as it has been presented in the literature, is that it apparently leads one to what we refer to as the "Paradox of Infinite Limits", which poses a challenge to scientific realism. Informally, the intended paradox can be formulated as follows. On the one hand, a scientific realist must believe that real physical systems are finite, as it is indeed suggested by some of our most successful background theories such as the atomic theory of matter and general relativity. On the other hand, she must believe that the content of scientific theories invoking infinite idealizations is true or approximatively true insofar as they are indispensable to recover empirically correct results. Allegedly, that calls scientific realism into question[2].

This paradox was first introduced by Callender (2001) in the context of phase transitions and then further discussed, for instance, by Butterfield (2011), Norton (2012) and Shech (2013) [3]. Recently, interesting attempts to generalize the problem for scientific realism have made by Baron (2019), Liu (2019) and Shech (2019). However, the extent to which such proposals provide a definite solution is still open. Indeed, it remains unclear under what conditions the assumption of infinite limits leads one to paradoxes of the above form. The present paper aims to offer a general formulation of the Paradox of Infinite Limits. In the attempt to show how it can be resolved, we elaborate a taxonomy of the different uses of infinite limits in physics, which is partially based on distinctions made by Norton (2012) and Godfrey-Smith (2009). Specifically, we point out that when being understood as approximations and abstractions, infinite limits do not pose any substantial problems to scientific realism. Yet, when they give rise to infinite idealizations they can actually lead one to the Paradox of Infinite Limits, depending

[1]It is important to point out that one can also "construct" infinite systems without taking any limits such as the assumption of an infinitely long cylinder or two dimensional systems. Although such examples deserved attention and have been discussed in the philosophical literature (e.g. (Earman 2017; Shech 2018)), we will restrict our analysis to infinite idealizations resulting from the use of infinite limits.

[2]For completeness, let us clarify that in the literature one can find at least three different versions of scientific realism: Explanatorianism, Entity realism and Structural realism (Chakravartty 2017). In this paper we consider a general characterization of scientific realism, leaving the question of how these particular versions can deal with infinite idealizations for future work. However, we point out that the Paradox of Infinite Limits is especially problematic for Explanatorianism (Kitcher 1993; Psillos 1999), which recommends a realist commitment with respect to those parts of our best theories that are indispensable to explaining empirical success.

[3]See also (Fletcher, Palacios, Ruetsche, and Shech 2019) for a recent collection of papers on infinite idealizations in science.

on whether the idealization is regarded as *essential* for the explanation of the physical phenomenon under investigation. We then argue that, even in the case of Essential Idealizations, there are ways of coping with the alleged incompatibility between infinite idealizations and scientific realism, which ultimately rely on empirical considerations.

We organize the paper as follows. In Section 2, we distinguish between idealizations, approximations and abstractions broadly-constructed. This distinction is partially based on Norton (2012)'s distinction between idealizations and approximations and on Godfrey-Smith (2009)'s distinction between idealizations and abstractions. In the following section, we formulate the general Paradox of Infinite Limits and we explain in greater detail the sense in which it raises a challenge for scientific realism. In particular, in sub-section 3.3 we state the condition of Empirical Correctness in precise terms, and then we proceed to analyze the various possible understandings of the use of infinite limits in physics. Specifically, in section 4 we develop the concept of approximation and argue that it is not problematic from a scientific realist perspective by means of concrete examples: in fact, the paradox does not arise in the case of Approximations without Idealizations, whereas it can readily disarmed in the case of Idealizations yielding Approximations. In Section 5, we address the use of infinite limits as essential idealizations, explaining why that would lead the scientific realist to a paradox. Yet, by focussing on the controversial example of first-order phase transitions, we show that there is available a procedure to dispense the infinite idealization "on the way to the limit", thereby avoiding a contradiction with the claim that real target systems undergoing the phenomenon to be explained are finite. Finally, in the last section 6 we address the case of continuous phase transitions to illustrate the use of infinite limits as abstractions, and we also explain why these limits would not raise any conflict with scientific realism.

2 Idealizations, approximations and abstractions

2.1 Preliminary concepts

In the scientific practice it is ubiquitous to, so to speak, "modify" the systems encountered in the world with the goal of making our theories computationally tractable, pedagogically accessible or explanatorily rich. The representation of real planes as frictionless planes in which objects can move uniformly and perpetually is a prototypical example of this practice, in which real systems are modified with the purpose of making the theory manageable. Looking at the specific ways in which real systems are represented in scientific theories, one can distinguish at least between three different strategies labeled by philosophers of science as *idealizations*, *approximations* and *abstractions*.

Although there is no consensus about the nature of scientific idealizations, most philosophers agree that *idealizations* involve a misrepresentation of a real system (i.e. the target system) driven by pragmatic concerns, such as mathematical tractability. Recent accounts provide a more precise characterization of idealizations by arguing that they always refer to imaginary systems considered to be analogues of the real-world systems of interest. For example, Godfrey-Smith (2009, p. 47) says "I will treat [idealizations] as equivalent to imagining the existence of a fictional thing that is similar to the real-world object we are interested in". In a similar vein, Norton (2012, p. 209) defines an idealization as "a real or fictitious system, distinct from the target system, some of whose properties provide an inexact description of some aspects of the target system." According to these authors, there will be an idealization only when there is reference to a novel (fictional) system, which has properties that resemble the properties of real-world systems. In other words, when the misrepresentation of the properties of a target system coincide with the exact properties of a fictional system.

An example of idealizations is the frictionless plane mentioned above. This fictional system was firstly introduced by Galileo to derive the equations of motion of an object moving down an inclined plane. Although no such planes exist in reality, they have proven to be extremely useful to predict the behavior of real world systems. Why can these fictional systems explain the behavior of concrete systems observed in the world, and how can we justify their use from a scientific realist perspective? The standard justification for idealizations of this kind is that they provide approximately true descriptions of real world systems, where approximation to truth is simply understood as a relation of similarity between the properties of fictional systems and the properties of real world systems (Godfrey-Smith 2009). It is also believed that these idealizations are dispensable, in the sense that it is possible in principle to de-idealize the theory by systematically eliminating distortions and by adding back to the model details of concrete systems (McMullin 1985)[4].

This characterization of idealizations puts us in position to distinguish them from other strategies of theory construction such as approximations and abstractions. For instance, according to Norton (2012), *approximations* are inexact descriptions of certain properties of the target system, which are given in terms of propositions expressed in the language of a theory, that do not need to correspond to the relevant properties of some other fictional system. In fact, the use of approximations does not require one to make reference to any new system different from the target systems. In this sense, one can say that approximations involve distortions or misrepresentations of the target system but, differently from the case of idealizations, these distortions are merely propositional. For Norton (2012), idealizations can be demoted to approximations by discarding the idealizing system and extracting the inexact description, but the inverse promotion will not always succeed. However, we will see that there seem to be cases of *essential idealizations* in which idealizations cannot be easily demoted to approximations.

In recent years, the topic of *essential idealizations* has generated a great deal of excitement among philosophers of science. In particular, it has been argued that "infinite idealizations" arising from mathematical limits that are ineliminable cannot provide approximations of realistic systems, because the latter exhibit behavior that is qualitatively different from the behavior exhibited by the idealized infinite systems (e.g. (Batterman 2005)). We will examine some possible cases of ineliminable idealizations in Section 5, but before doing so let us mention some examples of approximations and abstractions.

Let us recall that an approximation is an inexact description of the target system that does not (necessarily) involve the introduction of a fictional system. This means, there is no appeal to fictional system in which the inexact properties of the target system are true. A good example of an approximation without idealization also mentioned by Norton (2014) is the case of a mass falling in a weakly resisting medium. As we know, the speed v of a falling mass at time t is given by:

$$dv/dt = g - kv,$$

where g is the gravitational acceleration and k the friction coefficient. The speed of the mass at time t, as it starts from rest, is given in terms of the Taylor expansion series by:

$$v(t) = (g/k)(1 - exp(-kt)) = gt - gkt^2/2 + gk^2t^3/6 - ...$$

If we assume that the friction coefficient is small, we can approximate the previous expression by taking only the first term of the series expansion:

$$v(t) = gt$$

When we use this expression to describe the behavior of a real mass falling in a resisting medium, we do not intend to give a literal description of the situation, but rather to give a good approximation of it. This strategy allows us to simplify problems that may be otherwise untractable. It is important to note, however, that one can promote this approximation to an idealization by introducing a fictional system in which a body falls under the same gravity in a vacuum, so that the fall is described exactly by $v(t) = gt$ ((Norton 2014)).

So understood, idealizations can also be distinguished from abstractions. According to Godfrey-Smith (2009), an *abstraction* is the act of "leaving things out while still giving a literally true description of the target system". (p. 48). In contrast to idealizations, abstractions do not intend to state claims that are literally false and do not make reference to fictional systems.[5] Instead, they involve the omission of a truth by leaving out features considered to be irrelevant. Abstractions also differ from approximations

[4]This kind of idealization corresponds to what has been called "Galilean idealization" (e.g. (McMullin 1985; Weisberg 2007)).

[5]A similar view is defended by (Jones 2005)

in that the former do not involve propositional misrepresentations of the target system whereas the latter do.

2.2 Approximate truth in the context of idealizations, approximations and abstractions

We will argue next that all the forms of inaccurate representations mentioned in the previous section, i.e. idealizations, approximation and abstractions, can be made compatible with the more relaxed realist framework that accepts the content of scientific theories to be at least approximately true. However, in order to arrive at that conclusion, we need to offer a more precise definition of the notion of "approximate truth". Chakravartty (2010) distinguishes between three kinds of approaches for approximate truth in the standard philosophical literature: the verosimilitude approach, the possible-world approach and the type hierarchy approach. The verisimilitude approach, elaborated by Popper (1972), consists in comparing the true and false consequences of different theories. In the possible word approach, which is meant to be an improvement of the verosimilitude approach, the truth-likeness is calculated by means of a function that measures a mathematical "distance" between the actual world and the possible worlds in which the theory is strictly correct (Tichy 1976; Oddie 1986) so that one can generate an ordering of theories with respect to truth-likeness. Finally, in the type hierarchy approach, truth-likeness is calculated in terms of similarity relationships between nodes that represent concepts or things in the word (Aronson 1990). As Chakravartty (2010) points out, the problem that all these approaches have in common is that they do not pay attention to the different ways in which scientific representations give inaccurate account of the target systems. This is an important limitation of these approaches, because the notion of approximate truth is best understood differently in different circumstances, especially in cases of idealizations yielding approximations, essential idealizations, approximations and abstractions.

Let us discuss first the notion of "approximate truth" in the case of idealizations. As said above, idealizations involve a misrepresentation of a real system (i.e. the target system) by means of introducing an imaginary system considered to be an analogue of the real-world systems of interest. Here we follow (Chakravartty 2010, p. 40) in considering that the adequate notion of "approximate truth" concerns the degree to which this fictional system that successfully captures aspects of the target system resembles a non-idealized representation of that system. In some cases, especially when the idealization is a limit case of the de-idealized system, the degree of resemblance can be specified mathematically so that one can even quantify the degree of misrepresentation. All cases of idealizations yielding approximations can be put in this category and will be discussed in greater detail in the next sections. These cases do not represent a challenge for scientific realism since the notion of "approximate truth" can be adequately quantified.

A similar notion of "approximate truth" can be given in the case of approximation that do not involve idealizations. The difference is that degree of resemblance should not evaluated between the properties of a fictional system and those of a target system, but rather between the misrepresentation of the properties of the target system and the actual properties of the real target system. In cases like the example of a mass falling in a weakly resisting medium, we can quantify the degree of misrepresentation by considering the terms of the series expansion that have been taken into account. In fact, more accurate descriptions will imply incorporating more terms of the series expansion. The more terms we consider, the better the approximation of the real properties of the target system will be. In this sense approximations are straightforwardly compatible with realism, in that it is possible to quantify the degree of misrepresentation.

The challenge for scientific realism comes instead from the possibility of *essential idealizations*, which appear to be ineliminable to the explanation of a certain phenomenon and cannot be de-idealized towards more faithful explanations. In other words, they cannot be demoted to approximations. In these cases, there do not seem to be a straightforward notion of "approximate truth" and the degree of resemblance between the fictional system and the target system do not seem to be easily quantifiable. We will discuss such cases in section (5).

Finally, let us refer again to abstractions. As said above, abstractions omit details that are considered to be irrelevant but do not involve any misrepresentation of the target system. For example, in trying to describe the behavior of a cannonball that has been fired on some particular day, there are innumerable features that will not be taken into account such as the composition of the cannonball, its color, its temperature or the mechanism by which the initial velocity is conferred to the ball. The scientific model that predicts where the cannonball will land may include a number of distortions with respect to other properties like the gravitational force, which is generally assumed to have the same magnitude and direction at all points of the trajectory. However, such a model does not involve misrepresentation with respect to the properties that do *not* make a difference for the occurrence of the phenomena. In fact, the model is simply silent about them, and hence it does not say anything false as regards these irrelevant properties (Jones 2005). Although this strategy does not involve misrepresentation of certain properties it does give an inexact description of the target system because it leaves out factors that are irrelevant for the behavior under consideration. One then need an appropriate notion of "approximate truth" in this case too. The intended notion will differ from the one involved in cases of idealizations and approximations in that there is no misrepresentation of properties. Chakravartty (2010, p. 39) suggests an articulation of the notion of approximate truth *qua* abstractions connected with the notion of comprehensiveness: "The greater the number of factors built into the representation [i.e. the more comprehensive is the description], the greater its approximate truth". In so far as one can be quantify these factors, there will be a precise notion of approximate truth applying also to the case of abstractions. It is important to note that sometimes abstractions may be even crucial or essential for the explanation of certain classes of behavior. [6] For instance, if we want to explain the behavior of cannonballs in general, we should not include details that are specific of a certain fire. This is possible just because abstractions generally aid in the explanation of certain phenomena by enabling us to account for some common behavior that is generated among disparate systems (See also Weisberg (2007)).

3 The Paradox of Infinite Limits: A Challenge for Scientific Realism?

Mathematical limits are vastly used in physics. For instance, in the statistical mechanical theory of phase transitions, it is assumed that the number of particles as well as the volume of the system goes to infinity; similarly, in the ergodic theory of equilibrium and in the explanation of reversibility in thermodynamics it is assumed that time goes to infinity. Appealing to infinite limits has the technical advantage that they render the formal account of physical phenomena more tractable. Furthermore, in some cases, like in the examples we just mentioned, it even appears as a necessary condition to offer a mathematical description of real target systems in agreement with the empirical results. Nevertheless, when the variable growing to infinity corresponds to a physical parameter, e.g. the number of microscopic constituents, taking the limit introduces an unrealistic assumption, at least as long as real systems are believed to be finite. This raises many conceptual questions. For instance: How can we explain the empirical success of models that introduce such an unrealistic assumption? To what extent models that use infinite limits are compatible with scientific realism? In this section and the reminder of this paper we will address these questions based on the previous distinction between idealizations, approximations and abstractions.

3.1 A Taxonomy for Infinite Limits

The use of infinite limits has been frequently equated with the use of an infinite idealization. However, we want to stress a distinction between different uses of infinite limits as approximations, idealizations and abstractions. In order to draw such a distinction, we begin by setting up the formal framework within which our discussion is cast. Let S_n represent a system characterized by some physical parameter n, which may take on discrete or continuous values: in particular, it could denote the number N of molecules constituting a gas system, so that the parameter takes on values in the natural numbers $\mathbb{N}$; or,

[6] This account of abstraction is consistent with Cartwright's view, according to which an abstraction is a mental operation, where we "strip away – in our imagination – all that is irrelevant to the concerns of the moment to focus on some single property or set of properties, as if they were separate." (1994, p. 187)

it could denote the time t during which a physical process unfolds, so that the parameter takes on values in the real numbers $\mathbb{R}$. As the variable n increases, one defines the following sequence of systems

$$\{S_1, S_2, ..., S_n\}_{n \in \mathbb{N}, \mathbb{R}}$$

The limit of such a sequence for $n \longrightarrow \infty$, if it exists, corresponds to the infinite system S_∞. Arguably, the latter is just a mathematical entity, and as such it would represent a fictitious rather than a real system. If so, by recalling the content of the previous section, taking the limit where the variable n goes to infinity gives rise to an idealization. We can thereby characterize the limit system S_∞ as an *infinite idealization*.

Now, limits can also be used as approximations without idealizations. Consider again the above sequence of systems $\{S_1, S_2, ..., S_n\}_n$, we can also define the following sequence of functions representing a putative physical quantity f:

$$\{f_1, f_2, ..., f_n\}_{n \in \mathbb{N}, \mathbb{R}}$$

where the notation f_n indicates the relevant property possessed by each finite system S_n, so that $f_n := f_{S_n}$. Here, some care should be taken when evaluating the infinite limit $n \longrightarrow \infty$. In fact, as Butterfield (2011) suggested, there is a crucial difference between "the limit of a sequence of functions" and "what is true at that limit": that is, respectively,

- (i) the limit f_∞ of the sequence $\{f_1, f_2, ..., f_n\}_n$ of functions, and
- (ii) the function f_{S_∞} associated with the limit system S_∞.

More to the point, if the parameter n takes on values on the natural numbers, (i) obtains when one adjoints infinity to the set $\mathbb{N}$, and hence the limit $f_\infty := \lim_{n \longrightarrow \infty} f_n$ is defined as the function being the last element of the above sequence with $n \in \mathbb{N} \cup \{\infty\}$. In this case, there is no reference to the infinite system S_∞, so that using the limit in this sense does not lead to an idealization. To the contrary, (ii) depends exactly on how the limit system is constructed. Indeed, the limit function f_{S_∞} represents the relevant quantity possessed by the infinite system S_∞: as such, it tells us just what is true at the limit. It should be emphasized, though, that its existence is contingent upon the type of convergence one adopts: this point will be useful for our discussion in section 5 below.

The proposed distinction becomes less abstract if one casts it in terms of values of quantities. In fact, supplying numerical values is right what enables us to directly check whether or not the expected results turn out to be empirically correct. So, given that the putative function f takes on values $v(f)$, one should consider yet another sequence, namely the sequence of values

$$\{v(f_1), v(f_2), ..., v(f_n)\}_{n \in \mathbb{N}, \mathbb{R}}$$

Of course, the actual value of each function f_n depends on the state s_n in which the system S_n is: in fact, there is also a sequence of states $\{s_1, s_2, ..., s_n\}$ implicitly understood along with the sequence of systems. As above, one needs to distinguish between (i) the limit of the sequence of values of the function f, i.e. $\lim_{n \longrightarrow \infty} v(f_n)$, and (ii) the value $v(f_{S_\infty})$ of the natural limit function computed when the limit system S_∞ is in the limit state s_∞.

Before addressing the issue whether, and how, it is possible to dispense from the infinity system in the explanation of some physical phenomenon, let us conclude this section by noting that the formal setting we have just presented enables us to sharply distinguish between different ways to characterize the use of infinite limits in physics. The proposed taxonomy identifies three main types, that is:

1. *Approximations without Idealizations*, where (ii) is not well defined, or (i) is empirically correct but (ii) is not;
2. *Idealizations yielding Approximations*, where (i) and (ii) are well defined and equal; and

3. *Essential Idealizations*, where (i) and (ii) are well defined but are not equal, and (ii) rather than (i) is empirically correct.

Beside these cases, one can add to this taxonomy the use of *infinite limits as abstractions*, which does not directly follow from such a scheme: in fact, as we will see in greater detail in section 6, these are cases in which the variable n does not represent any physical parameter of the target system. For example, the parameter can represent the number of times that one has to apply a transformation that successively coarse grains the system. Here the appeal to an infinite limit is merely instrumental in that it allows us to find fixed points in a topological space.

In the rest of the paper, we evaluate how each type of infinite limit fares against the so-called Paradox of Infinite Limits and its consequences for Scientific Realism, which we present here below.

3.2 The Paradox of Infinite Limits

Suppose that in order to represent a physical phenomenon P we define a system of the form S_n, then the Paradox of Infinite Limits can be essentially formulated as a combination of the following statements:

- (I) *Finiteness of Real Systems*: If S_n represents a real system, then the variable n corresponding to some physical parameter cannot take on infinite values.
- (II) *Indispensability of the Limit System*: The explanation of the phenomenon P can *only* be given by means of claims about an infinite system S_∞ constructed in the limit $n \longrightarrow \infty$.
- (III) *Enhanced Indispensability Argument (EIA)*: If a claim plays an indispensable role in the explanation of a phenomenon P we ought to believe in its existence.

The ostensive problem can be further articulated and made more precise when dealing with the description of particular phenomena, as we will see in greater detail for the Paradox of Reversible Processes and the Paradox of Phase Transitions in sections 4 and 5, respectively. But the tension between the above statements captures the core idea of the paradox. Intuitively, the fact that real systems are finite in the sense expressed by statement (I) can be understood as a basic desideratum of scientific realism. Indeed, according to our most successful theories, such as the atomic theory of matter, real gases contain only a finite, albeit extremely large, number N of molecules, just as real thermodynamical processes, even when being very slow, always take a finite amount of time t to complete. However, if one commits to the reality of an infinite system as demanded by statement (II) together with statement (III) then one infringes on such a basic desideratum. In fact, a seeming contradiction with statement (I) arises insofar as appealing to the infinite idealization S_∞ proves necessary. In their strongest form, claims that taking the limit is indispensable are backed by no-go theorems, established within the formalism of a given theory, to the effect that certain features of P cannot be recovered unless n is infinite. A threat to scientific realism can then be mounted when these results are coupled with statement (III), which Baker (2009) called Enhanced Indispensability Argument (EIA). According to the EIA, we ought to rationally believe in the existence of claims (e.g. mathematical entities or idealizations) that play an indispensable role in our best scientific theories (cfr. (Shech 2019) for a discussion of EIA in relation to infinite idealizations). In fact, it follows that, if the best explanation available for the physical phenomenon P requires one to take the limit $n \longrightarrow \infty$, owing to EIA one is bound to ontologically commit to the infinite system S_∞, even though statement (I) entails that the latter cannot correspond to any real system. A realist stance towards our best physical theories is thus endangered by the Paradox of Infinite Limits.

In order to resolve the problem, one should jettison one of the three statements that jointly engender the ostensive paradox, at least in the form they have been presented above. The first statement seems rather uncontroversial, and as such it is hard to give it up. Indeed, the finiteness of a real system S_n involved in the phenomenon P to be explained is grounded in empirical considerations, so long as the variable n denotes a physical parameter. Furthermore, in some cases the truth of statement (I) is granted by the content of successful background theories: for example, the number of molecules in a thermal system being finite is presupposed by the atomic theory of matter itself and general relativity (See Baron (2019)). Hence, insofar as the variable n represents a physical parameter as in statement (I) in the paradox, the culprit must be traced back to statement (II) or statement (III). Our own strategy will consist in giving up statement (II), but let us first review some attempts to block statement (III).[7]

Baron (2016) casts doubts on statement (III) by offering a helpful characterization of the sense in which idealizations can be regarded as indispensable to the best explanation of a phenomenon. According to him, although idealizations may be indeed indispensable to the purported explanation, they do not carry explanatory load (understood as counter-factual dependence between the idealization and the effect) and therefore should not be reified. However, this strategy is questionable in the case of infinite idealizations, since the latter can have explanatory load, at least in the sense that we will expose in section 5. In (Baron 2019), he adopts an alternative strategy by distinguishing between constructive indispensability and substantive indispensability: given a mathematical entity that is explanatorily indispensable to our current best scientific theories, according to the former notion there is reason to believe that such a claim can actually be dispensed, although we do not know how to do it yet; according to the latter notion, instead, there is no reason to suppose that the claim can ever be dispensed. In other words, in the first case indispensability is just a contingent matter, whereas in the second case it is an unavoidable fact. So, under this understanding, if an infinite limit that appears as necessary to explain a physical phenomenon P is constructively rather than substantively indispensable, the Enhanced Indispensability Argument does not apply with sufficient cogency to commit one to the reality of the infinite system constructed in the limit, thereby disarming the implications of the Paradox of Infinite Limits for scientific realism. The crucial question then becomes how to determine whether one is dealing with an instance of substantive indispensability or an instance of constructive indispensability. Although the exact answer can only be given on a case-by-case basis, there are two major strategies that one may adopt, according to Baron. One approach, which Baron himself favors, is grounded on the notion of coherence. Specifically, one ought to test the alleged indispensability of an infinite limit against other background scientific theories: if the claim under test is not consistent with other accepted theories, then there is reason to supposed that it is only constructively indispensable. For instance, when a gas system is contained in a finite region the limit for the number N of molecules going to infinity is at odds with the atomic theory of matter; moreover, Baron argues, it fails to cohere with the general theory of relativity in that the total mass of the molecules would become infinite despite the volume remaining finite, and hence the density would result infinite thereby giving rise to a black hole in the region where the gas is confined, which is not really observed in real-life phenomena. Although plausible, this solution seems to beg the question, since the problem that we are trying to solve concerns precisely the inconsistency between well established background theories and mathematical entities used to explain certain phenomena. For us, the motivation for accepting statement (III) comes from the acceptance of the inference-to-the-best-explanation. Indeed, the best argument for scientific realism is the No Miracles Argument, which maintains that we ought to believe that abstract and theoretical claims about existing entities postulated by our most successful theories are (at least approximately) true because otherwise their success would be a miracle. Underlying this argument is the inference-to-the-best-explanation (Boyd 1983). By using a similar reasoning, if the best explanation available of a certain phenomenon P involves assuming an infinite idealization and there is no known way to de-idealize the model without losing explanatory power, then we should commit to the existence of such an idealization for the same reasons that ground scientific realism. This a particularly important for Explanationanists (Kitcher 1993; Psillos 1999), since they recommend holding a realist attitude towards entities that are indispensable to the explanation of certain phenomena. [8]

[7]See (Colyvan 2001), (Baker 2005; Baker 2009) and (Baron 2016) for a careful discussion of the indispensability arguments.

[8]Shech (2013, p. 1177) makes a similar point, when he says:

> Insofar as arguments like the “no miracles argument” and “inference to best explanation” are cogent and give us good reason to believe the assertions of our best scientific accounts, including those about fundamental laws and unobservable entities, then in the case of accounts appealing to [essential idealizations], these arguments can be used via an Indispensability Argument to reduce the realist position to absurdity.

Therefore, we claim, a more satisfactory solution to the Paradox of Infinite Limits comes from the rejection of statement (II), rather than statement (III). To make our point, we need a criterion of empirical correctness that we introduce next.

3.3 The Condition of Empirical Correctness

Before spelling out our intended criterion for correctness, it should be stressed that the concept of explanatory indispensability heavily depends on what one means by scientific explanation, which is a huge and much debated topic in philosophy of science.[9] For our purposes, we restrict ourself to what we conceive as a minimal requirement for a good explanation of some physical phenomenon P, namely that one recovers empirically correct results. Arguably, this yields a necessary condition in that, if an account fails to agree, at least approximatively, with the observed data then it cannot be said to explain P (whether it yields also a sufficient condition is less straightforward to establish, but our argument here does not really need that much). The relevant data are given by the values of physical quantities of interest, like energy, position, momentum, spin, etc., corresponding to properties of the physical system involved in the phenomenon. The proposed condition for empirical correctness is as follows:

> **Empirical Correctness**: Let D be the observed data relative to a physical quantity represented by the function f, which characterizes the physical phenomenon P: then, the system S_n recovers *empirically correct results* for f just in case $v(f_n) \approx D$, in the sense that there exists an arbitrarily chosen real number $\varepsilon > 0$ such that $|v(f_n) - D| < \varepsilon$.

Let us explain the content of this definition. Note that one cannot expect that empirical data can be sharply determined, in that observations always involve some margin of error. Hence, our condition is formulated in such a way to allow for degrees of inexacteness: in fact, it only requires that the value of f be approximately equal, rather than exactly equal, to the data D, that is $v(f_n) \approx D$ for some given n. Deciding the degrees of inexactness that one may tolerate is ultimately a pragmatic matter, and that is why the number ε is left unfixed in the definition: yet, by keeping ε very small, one ensures that the results produced by system S_n are empirically correct to a pretty good approximation.

Infinite limits can be used in the explanation of the physical phenomenon P when empirical correctness is satisfied by the infinite system S_∞ for some physically relevant function f. Accordingly, one has $v(f_{S_\infty}) \approx D$, meaning that the value at the limit, namely (ii) the function f_{S_∞} associated with the limit system S_∞, is the same, or at least approximatively the same, as the observed data D. More to the point, infinite idealizations are regarded as indispensable to the explanation of P insomuch as no finite system S_n can yield empirically correct results for f. That is, even though the parameter n grows while still remaining less than ∞, the values taken on by f_n fail to provide an approximation of the observed data for some sufficiently small ε. To put it in technical terms, this alleged indispensability of the infinite idealization typically manifests itself in the cases in which the limit is singular, and hence (i) the limit $\lim_{n \longrightarrow \infty} v(f_n)$ of the sequence of values does not coincide with (ii) the value $v(f_{S_\infty})$ yielded by the limit system (Butterfield (2011) actually makes a similar point when discussing the emergence of certain properties at the limit).

Armed with the above-stated condition of empirical correctness, we now proceed to show how infinite idealizations can be actually dispensed to the explanation of physical phenomena that seem to require one to take the limit $n \longrightarrow \infty$, thereby allowing one to give up statement (II) in the Paradox of Infinite Limits. In fact, such a condition grounds the use of limits as approximations of the properties of real target systems. As we argue below, that is the basis to demonstrate that cases of idealizations yielding approximations as well as the more controversial cases of essential idealizations do not pose a threat to Scientific Realism.

[9] See (Woodward 2014) for an excellent review of the different approaches to scientific explanation in the philosophical literature.

4 Approximations with and without Idealizations

In section 2 we gave a general characterization of the concept of approximations as inexact descriptions of target systems. Here, we can make this idea more precise thanks to the framework presented in previous section, according to which the relevant properties of a target system corresponding to physical quantities are represented by mathematical functions yielding numerical values: in fact, the very condition of empirical correctness rests on the possibility that such values match the observed data with a certain margin of accuracy. An approximation can therefore be defined as a formal description of some specific property of the target system that, despite being inexact, puts one in a position to recover empirically correct results. The choice of what properties are relevant to the purported description as well as the degrees of inexactness being allowed are dictated by pragmatic considerations regarding the physical phenomenon to be explained. Such an understanding of approximations is particularly suitable for our discussion of the use of mathematical limits, the more so because it is probably the most common attitude physicists tend to take on in their scientific practice. For, recall that in order to assuage the worries concerning scientific realism posed by the Paradox of Infinite Limits one ought to find a way to dispense from the infinite idealization introduced in statement (II) of the paradox. Interpreting infinite limits in terms of approximations, as contemplated by the first two classes listed in our proposed taxonomy, enables us to do so. Let us discuss both scenarios in general terms, and then apply our reasoning to particular examples.

The scenario in which the limit $n \longrightarrow \infty$ gives rise to an *idealization yielding approximations* may appear complicated in that it features a fictitious infinite system S_∞ that gives an inexact description of the real target system, and as such it effectively satisfies the condition of empirical correctness. However, given that in this case the limits (i) and (ii) in Butterfield's above distinction coincide, there is a strategy to dispense the infinite idealization that is readily available. Specifically, one can show that, as the variable n grows, for some finite value $n_0 < \infty$ the behaviour of the corresponding system S_{n_0} is approximatively the same as the behaviour of the limit system S_∞, thereby recovering empirically correct results for the relevant properties "on the way to the limit" rather than at the limit. More precisely, n_0 is supposed to be the actual value characterizing the real target system that undergoes the physical phenomenon P to be explained: so, if the value of the function $f_{S_{n_0}}$ is sufficiently close to the limit value of f_{S_∞}, one does not need to ontologically commit to S_∞ in order to fulfill empirical correctness. One can thus, in the same way as we presented above, dispense the infinite idealization to the explanation of P, in full compliance with statement (I) of the Paradox asserting the finiteness of real systems. If idealizations yield approximations, though, one may still wonder why one should appeal to a mathematical limit in the first place. In other words, one may ask, how can one justify the use of an infinite limit to describe the target system? In order to answer this question, Butterfield (Butterfield 2011) puts forward what he calls a Straightforward Justification, which is based on two features that limits enjoy, namely mathematical convenience and empirical adequacy. Regarding the first feature, taking the limit often enables one to ignore some degrees of freedom that complicate the calculations, and so infinite systems, when they exist, turn out to be more tractable than finite systems for which the parameter n is very large[10]. As for empirical adequacy, arguably that assures that the values obtained at the limit are close enough to the real values. Hence, while the empirically correct results are given by the values of the function computed for the actual n_0, namely the values of the real target system, taking the limit for n growing to infinity still proves adequate within some acceptable margin of approximation. Thus, based upon these two desirable features proposed by Butterfield, one has both pragmatic and empirical reasons to justify the use of the infinite limit. That reinforces our claim that one does not need to commit to the reality of the infinite idealization S_∞, contrary to what is entailed by statements (II) and (III) in the Paradox of Infinite Limits.

The other scenario, namely the case in which taking the infinite limit $n \longrightarrow \infty$ yields an *approximation without idealization*, is a more straightforward to deal with. On the basis of our taxonomy, it

[10]In addition, (Palacios 2018) points out that it does not suffice to show that the behavior that arises in the limit arises already for a large value of the parameter n, but one also needs to show that it arises for realistic values of n_0. This latter requirement ought to be empirically grounded and is meant to assure that the lower level theory that results from a limiting operation is empirically correct and therefore capable of describing realistic behavior.

occurs when the limit (i) rather than (ii) yields empirically correct results or (ii) is not well-defined at all for some physically significant function f: as a consequence, the infinite system S_∞ does not satisfy empirical correctness. It follows that it cannot be even used to explain the relevant phenomenon. Indeed, an approximation without idealization should be understood as a misrepresentation of the target system whose properties are given an inexact description in terms of the limit f_∞ of the sequence $\{f_1, f_2, ..., f_n\}_n$ of functions representing the relevant physical quantities, rather than in terms of the properties of a fictional limit system. Accordingly, it would not make any sense to reify the infinite idealization, which means that statement (II) in the Paradox of Infinite Limits does not hold and hence there cannot arise any ensuing problem for scientific realism. Very recently, Norton (2012) recognized that sometimes infinite limits are used precisely as approximations without idealizations. For him, there are two sufficient conditions under which an approximation cannot be promoted to the status of idealization, which are closely related to the formal conditions we stated above: that is, (1) the limit system does not exist in the sense that it is paradoxical, and (2) the limit system has properties that are inadequate for the idealization in that they do not match with the properties of the finite target system. The following quotation illustrates his proposed criterion for the failure of idealizations:

> Another type of limit used in thermodynamics cannot be used to create idealizations. Its limiting processes are beset with pathologies so that it either yields no limit system or yields one with properties unsuited for an idealization. (Norton 2012, p. 13)

Norton's goal is to show that, on the basis of these two conditions, some infinite limits that are regarded as idealizations in the literature do not deserve to be elevated to such a status and should in fact be demoted to mere approximations, as it happens in many examples in which mathematical limits are employed in thermodynamics and statistical mechanics[11]. In his view, an example in which an infinite limit does not give rise to a suitable idealization due to condition (2) is the account of phase transitions in the thermodynamical limit within classical statistical mechanics that we will address in the next section. Another controversial case that Norton claims to be an instance of failure of idealization due to condition (1) is the paradox of thermodynamically reversible processes arising in the infinite-time limit, which has recently drawn attention in the philosophical literature (Norton 2014; Norton 2016; Valente 2017). Let us discuss this case next.

Reversible processes are conceived as sequences of equilibrium states through which a thermodynamical system progressively passes in the course of time. They are typically constructed by means of the infinite-time limit, and as such they are interpreted as processes taking place infinitely slowly. They owe their name to the fact that, ideally, they could be traversed in both directions of time. However, reversible processes are fictitious processes: indeed, thermodynamical processes just take a finite amount of time t to complete, even if they proceed quasi-statically, and they can only unfold in one temporal direction but not in the reversed one. So, at best, taking the infinite-time limit $t \longrightarrow \infty$ can yield approximations of real processes. The question that interests us is whether or not the limit processes can also be intended as infinite idealizations. According to Norton, reversible processes do not deserve to be elevated to such a status since they display contradictory properties. More to the point, he submits that they are plagued by a paradox: for, on the one hand, (I) thermodynamical processes are driven by a non-equilibrium imbalance of driving forces, which is necessary in order to enact the transition of the system from one state to another; on the other hand, though, (II) a reversible process amounts to a sequence of equilibrium states, for which there must be no imbalance of driving forces. By connecting the driving forces enacting the process with its duration in time, Norton's Paradox of Reversible Processes becomes somewhat similar to the general Paradox of Infinite Limits. For instance, when a fixed quantity Q of heat is exchanged between two bodies in thermal contact, the driving forces are given by

[11] To be sure, Norton concedes that, under certain circumstances, approximations can actually be promoted to idealizations. For instance, in his (2014) paper Norton presents the case of the law of ideal gases in thermodynamics as an example of approximation which gives rise to an ideal system that can as well serve as an idealization. In fact, an ideal gas system is defined as constituted by a large number of non-interacting, spatially localized particles, and as such it can describe, at least approximately, the behaviour of very rarefied real gases under appropriate circumstances.

the temperature difference ΔT between the bodies. So, if the latter is equal to zero, the process of heat transfer cannot take place. Now, by assuming Fourier law $Q = -k\Delta T \Delta t$ (with k being the heat transfer coefficient), the amount of heat that passes from the hotter to the colder body is also proportional to the time Δt that the process would take to complete. As a consequence, if one tries to construct a reversible process by letting time go to infinity, the temperature difference must vanish, as prescribed by statement (II), but then no heat would be exchanged between the two bodies, thereby raising a contradiction with statement (I). Based on such a paradox, Norton argues that reversible processes fail to be idealizations.

However, (Valente 2017) objected to this conclusion, by offering a way to circumvent the alleged contradiction. As he pointed out, Norton's paradox arises due to the misconception that reversible processes should be treated as actual thermodynamical processes. Instead, they ought to be regarded as mere mathematical constructions that are introduced to apply infinitesimal calculus to thermodynamics. In fact, formally, they simply correspond to continuous curves in the space of equilibrium states of a thermal system: along these curves one can calculate the exact integrals of state-functions, such as entropy, and then compute the values of other physical quantities of interest, like heat and work. For this reason, as it was observed by the mathematical physicist Tatiana Afanassjewa (1956) in her book on the Foundations of Thermodynamics, reversible processes should better be called "quasi-processes", so as to avoid the misunderstanding that they would correspond to actual processes, which could even be reversed. Accordingly, while statement (II) holds by definition, it would be a mistake to ascribe to reversible processes the properties required by statement (I): that is, contrary to what happens during real thermodynamical processes, there cannot be any non-equilibrium imbalance of forces moving a thermal system from one state of equilibrium to another along the continuous curve representing a quasi-process. As a result, the ostensive paradox disappears, and hence one may as well regard reversible processes as idealizations, in accordance with Norton's own criterion. Notice that here, differently from the Paradox of Infinite Limits, in order to disarm the apparent contradiction we need to drop statement (I) rather than statement (II): the reason is that what is to be explained in this case is just the idealized object constructed in the limit. The connection with real thermodynamical processes can then be given thanks to the notion of approximations. To illustrate this idea, let us again refer to Afanassjewa's own work. After warning that, strictly speaking, infinitely slow processes cannot exist (cfr. p.11), she went on to argue as follows:

> In order to make a quasi-process [i.e. a reversible process] open to experimental investigation or just to connect it with experiments in thought, one has to conceive it as approximated by quasi-static processes [i.e. extremely slow processes]. These are in turn real processes, if also idealized. (Ehrenfest-Afanassjewa 1956, p. 56)

Thus, even though an infinite-time limit process has no counterpart in reality, in practice it can serve to describe, albeit inexactly, the relevant properties of thermodynamical processes unfolding very slowly. To put it in terms of our characterization of idealizations yielding approximations: once a certain margin of accuracy is stipulated, the fictitious quasi-process constructed in the limit $t \longrightarrow \infty$ gives approximatively true descriptions of real physical quantities, such as heat and work exchanged during real processes that take a large yet finite time t_0 to complete.

So, although Norton's distinction between approximations and idealizations may be fruitful, the extent to which the examples he addresses are genuine cases of approximations without idealizations is still subject to debate. In our opinion, a less controversial case where one can have approximations without idealization is the reduction of the classical equations of motion to relativistic equations in the Newtonian limit. As it is well known, in the theory of special relativity physical quantities such as the energy and momentum of a moving body with an invariant mass m_0 can be expressed in terms of the so-called Lorentz factor γ:

$$E = \gamma m_0 c^2$$

$$p = \gamma m_0 v$$

Arguably, when the Lorentz factor goes to 1, one can recover the values of the classical counterparts of these quantities defined, respectively, as $E = m_0c^2$ and $p = m_0v$. Since the Lorentz factor takes on the form

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}},$$

one can make this expression go to 1 in the limit $(v/c)^2 \longrightarrow 0$, where c is the speed of light and v the velocity of a moving body in a given inertial frame. There are different possible ways of interpreting this limit. First, one can interpret it as taking the limit of a sequence of systems in which the velocity v goes to zero. The limit will then give rise to an idealization, namely to a fictional system in which the value of the velocity for all bodies is null in the given inertial frame. Nonetheless, understanding the limit in this sense has the problem that, e.g., the momentum p, which depends exactly on the velocity of the bodies, will be always zero. This naturally means that the quantities defined in the limit system will not provide a good approximation for the behavior of restless objects, in which momentum is different from zero. Therefore, the purported idealization would not serve to describe the behavior of moving objects. Likewise, a similar problem arises if one interprets the Newtonian limit as taking the limit of a sequence of systems in which c goes to infinity. In this case, the limit system will be an imaginary system in which the speed of light is infinite. In this system, quantities such as the kinetic energy that are defined as a function of the speed of light c will also go to infinity, thereby failing to give an approximation of the actual kinetic energy of real moving bodies, which is instead finite.

On the contrary, a much more suitable interpretation of the Newtonian limit $(v/c)^2 \longrightarrow 0$ should not be given in terms of fictional systems, but rather as a mere approximation of the velocity of bodies that are moving slowly compared to the speed of light. This can be seen more clearly by noticing that the Lorentz factor γ can be expanded into a Taylor series:

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}} = \sum_{n=0}^{\infty} \left(\frac{v}{c}\right)^{2n} \prod_{k=1}^{n} \left(\frac{2k-1}{2k}\right) = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{5}{16}\left(\frac{v}{c}\right)^6 + ...$$

Accordingly, if one considers only the first term of this Taylor expansion, one will recover the exact values of the classical quantities of interest, and this will constitute just an approximation of the velocity v of the objects for which $v << c$. Taking more terms into account will give results that are more accurate from the relativistic point of view, but that depart from the classical values. It is important to stress that the thus-described process of approximation is analogous to the one employed to account for the behavior of a mass falling in a weakly resisting medium presented in section 2, and therefore it lends itself to the same interpretation with respect to the notion of "approximate truth". In fact, limits of this kind should be understood as a mere misrepresentation of the values of certain properties of the target system, which will gradually disappear if one takes more terms of the series expansion into account. Note that here, like in all other examples of approximations without idealization, one does not posit any fictional system that would yield empirically adequate results, and hence the Paradox of Infinite Limits does not arise. Since there are no infinite systems involved, the procedure we have just outlined offers a straightforward way to make the process of taking the Newtonian limit compatible with a form of scientific realism that allows for theories to be approximately true.

In the last analysis, using mathematical limits to provide approximations of properties of real finite systems enables one to dispense limit systems from the explanation of the relevant physical phenomena. Indeed, as we have argued throughout the present section, despite the fact that letting some physical parameter grow to infinity is tantamount to introducing a *prima facie* unrealistic assumption, cases of approximations without idealizations and even cases of idealizations yielding approximations should not pose any worry to a scientific realist. A more outstanding challenge is instead raised by cases in which an infinite limit is used as an essential idealization, to which we now turn.

5 Essential Idealizations

The claim that there are *essential idealizations* has been put forward by some authors, especially Batterman (e.g. (2001, 2005, 2011)), in reference to singular limits, whereby empirically adequate results are obtained just for the infinite system and not for finite systems. In terms of Butterfield's distinction, this apparently mysterious case occurs when, given a function f representing some physical quantity, (i) the limit of the sequence of values and (ii) what is true at the limit for $n \longrightarrow \infty$ differ from each other, but it is only (ii) is empirically adequate (these cases correspond to *essential idealizations* according to the taxonomy presented in Section 3.1). So, if the variable n growing to infinity represents some physical parameter, it means that the limit system would not yield an approximation of the relevant property of the target systems, which are instead finite. An infinite idealization is then deemed as essential in that, arguably, it is only the limit system S_∞ that allows one to explain the physical phenomenon for which the function f is relevant. So, if one further assumes EIA, then one falls into the Paradox of Infinite Limits, hence threatening scientific realism. In this section we present a possible solution to this paradox that casts doubt on the "essential character" of the idealization, which is partially based on Butterfield's (2011) results.

5.1 Resolving the Paradox for First-order Phase Transitions

Phase transitions are sudden transformations of a thermal system from one state into another, occurring for instance when some material changes from solid to liquid state due to an increase of temperature. That is a much debated case-study where, according to indispensabilists like Batterman, there arises an essential idealization when taking the so-called thermodynamic limit. Informally, the argument goes as follows. According to thermodynamics, which deals with the behaviour of thermal systems from a macroscopic point of view, phase transitions occur when the function representing the derivatives of the free energy is discontinuous. Allegedly, such a discontinuity matches with the observed data, since sudden transitions from one phase to another are appear to take place abruptly. Yet, when attempting to recover the same phenomena within statistical mechanics, which describes thermal systems at the microscopic level as being composed by a very large number N of molecules, one faces a technical impossibility: that is, if N is finite, the function representing the derivative of the free energy remains continuous no matter how large N is. Instead, one can recover the sought-after discontinuity by taking the thermodynamical limit, which prescribes that both the number of molecules N and the volume V of the system go to infinity while keeping its density fixed (Goldenfeld 1992). This motivates the persistent attitude among physics, such as Kadanoff (2009), to emphasize the importance of infinite systems to explain the phenomenon of phase transitions:

> Phase transitions cannot occur in finite systems, phase transitions are solely a property of infinite systems (Kadanoff, 2009, p. 7)

Accordingly, it seems that one is bound to assume the infinite system S_∞ constructed in the thermodynamical limit as being indispensable in order to account for thermodynamical phase transitions within statistical mechanics. Hence, the thus-defined infinite idealization is supposed to be essential. If so, though, the Paradox of Phase Transitions would present itself.

In the philosophical literature, the paradox was originally proposed by Callender (2001). In his formulation, a contradiction arises due to the joint combination of four conditions: (1) Phase transitions are governed by classical statistical mechanics, (2) real systems have finite N, (3) phase transitions occur when the partition function has a discontinuity and (4) real systems display phase transitions. Let us stress that this is just a special case of the general Paradox of Infinite Limits. In fact, Callender's conditions can be explicitly connected with the three statements presented in Section 3.1. For, condition (2) is tantamount to statement (i) expressing the basic desideratum for scientific realism that real systems are finite, whereas condition (4) assures that the phenomenon to be explained, namely phase transitions, occurs for such systems, which is captured by statement (ii) . On the other hand, condition

(3) requires empirical correctness to be satisfied for the partition function being discontinuous: yet, as discussed above, if one tries to explain the phenomenon within the framework of statistical mechanics, in accordance with condition (1), then one is bound to reify the thermodynamic limit, which gives rise to an infinite idealization proving indispensable, exactly as our statement (iii) dictates. Other formulations of the paradox have been put forward in the literature, which differ from the one just presented only regarding how the allegedly conflicting conditions are stated, but they all basically agree on the content of the problem. There is also a variety of proposed solutions. Callender himself suggests that one should not take "too seriously" the theory that prompts one to introduce a discontinuity in the description of phase transitions, namely thermodynamics: accordingly, condition (3) can be dropped, which means that one does not need to appeal to the thermodynamical limit and therefore denies statement (ii) of our formulation of the Paradox. Shech (2013), on the other hand, suggests that one can resolve the paradox by noticing that the terms related to the existence of infinite systems do not refer to concrete physical systems but just to mathematical objects, which do not carry any ontological significance. However, as he correctly recognizes, this cannot be the attitude of scientific realists, who are interested in that our abstract scientific accounts gets something right about the real world. Alternatively, Liu (2019) suggests that the alleged contradiction disappears if one adopts a form of contextual realism, whereby realist claims should be evaluated relative to anchoring assumptions formulated within the background theory: in particular, the claim that the number of molecules grows to infinity can be regarded as true in the context of a microscopic theory holding that condensed matter is continuous. However, the jury is still out as to whether these proposals effectively dissolve the paradox of phase transitions.

Instead, our preferred strategy to elude the paradox and thus salvage scientific realism goes along the lines of (Butterfield 2011)'s own dissolution of the mystery of singular limits, which is endorsed, at least in connection with classical phase transitions, by (Menon and Callender 2013) as well as by (Norton 2014) and Palacios (2019). It develops into two steps: first of all, one ought to make a careful choice of the physical quantities to work with, so as to avoid those quantities for which the infinite idealization appears essential in that the limit is singular; then, one employs the notion of approximation to show that for the selected quantities empirically correct results obtain on the way to limit, without having to commit to the reality of the infinite system. Thus, the underlying idea of the proposed strategy is that of focussing on just the properties that are relevant for the behavior that we intend to explain, instead of requiring that all properties of the finite target system extend smoothly to the limit system. More to the point, Butterfield observes that the alleged mystery of singular limits is simply a consequence of looking at functions that do not give information regarding the behaviour of real systems as N increases towards infinity. For instance, if one restricts one's attention only to the fact that some function f be discontinuous, like in the case of the derivative of the free energy phase, one would lose insight of what happens for very large but finite N, since the expected discontinuity obtains only at the limit "$N = \infty$". To the contrary, one ought to turn one's attention to physical quantities represented by different functions, so that the corresponding properties of the actual target system S_{N_0} are approximated by the properties of S_∞. In other words, even though it appears to be an essential idealization for some quantities, the limit system is an idealization yielding approximations for other properties that one regards as physically salient for the phenomenon to be explained. In this way the desired behaviour is recovered on the way to the limit, and hence one can dispense from the infinite idealization, by demonstrating that this apparent "essential idealization" is a case of idealizations yielding approximations.

In order to make this proposal more concrete, let us look at a specific example of first-order phase transitions, that is a ferromagnet at sub-critical temperature. The Ising model portrays a ferromagnet as a chain of N spins, wherein a physical quantity called magnetization is represented as a function of the applied magnetic field. At very low temperatures, there are two possible phases available: a state where all spins are oriented in the up-direction, for which the magnetization takes on the value $+1$; and a state where all spins are oriented in the down-direction, for which the magnetization takes on the value -1. When the sub-critical temperature is reached, even if the applied magnetic field is null, one observes an abrupt flip from one phase to the other. That is formally captured by the magnetization function being discontinuous. Arguably, one can describe this phenomenon just in case one takes the thermodynamical limit whereby the number N of spins grows to infinity. Butterfield's proposed strategy can then be illustrated by means of a toy model. He defines a sequence of real-valued functions $\{g_1, g_2, ..., g_N\}_N$ with argument x belonging to the real numbers $\mathbb{R}$, which take on the following form:

$$g_N(x) := \begin{cases} -1 & \text{iff } x \leq -\frac{1}{N} \\ Nx & \text{iff } -\frac{1}{N} \leq x \leq \frac{1}{N} \\ +1 & \text{iff } x \geq \frac{1}{N} \end{cases}$$

All such functions are continuous, in that they remain equal to -1 until $x = -\frac{1}{N}$ and then they start to grow linearly with gradient N up to the value $+1$ for $x = \frac{1}{N}$, after which they remain constant again. However, when one takes the limit for $N \longrightarrow \infty$, the resulting function is no more continuous: in fact, the limit is given by

$$g_\infty(x) = \begin{cases} -1 & \text{iff } x < 0 \\ 0 & \text{iff } x = 0 \\ +1 & \text{iff } x > 0 \end{cases}$$

and hence it exhibits a discontinuity at point $x = 0$. Concretely, in the example of ferromagnetism, the behaviour of the functions g_N's mimics the magnetization function for a chain of N spins, with the argument x representing the applied magnetic field. A singular limit arises if one introduces a binary function $f_N : \mathbb{N} \cup \{\infty\} \longrightarrow \{0, 1\}$ whose numerical values are determined by whether g_N is continuous or not, i.e.

$$f_N := \begin{cases} 1 & \text{iff } g_N \text{ continuous} \\ 0 & \text{iff } g_N \text{ discontinuous} \end{cases}$$

Here, in accordance with the condition for essential idealizations stated in section 3 , when N goes to infinity one distinguishes between two possible values that are quite different: that is, (i) the value 1 taken on by all the functions in the sequence $\{f_1, f_2, ..., f_N\}_N$ for finite N, which results from the fact that all f_N are continuous; and (ii) the value 0 taken on by the function f_{S_∞} evaluated on the limit system S_∞, which results from the fact that g_∞ is discontinuous. So, if one focuses on the function f_N, the values for finite N's will always be continuous from the value of the quantity evaluated on the limit system, no matter how large N is. In particular, no real system with a finite number N_0 of molecules would have values of f_{N_0} that are approximated by the function f_{S_∞}. For Butterfield, that is just what makes the limit $N \longrightarrow \infty$ seem mysterious.

However, the mystery can be explained away if one looks at the behavior of the functions g_N's instead of the functions f_N's. In fact, as N grows, the functions g_N become more and more similar to the step function g_∞, even though, contrary to the latter, they remain continuous. Specifically, when N_0 is extremely large, for most points x one has $g_{N_0}(x) = g_\infty(x)$, which is true in particular when the argument is $x = 0$; furthermore, whenever the values of these functions are not strictly equal, namely around the singular point $x = 0$, they will still be close to each other, so that $g_{N_0}(x) \approx g_\infty(x)$. Therefore, the empirically correct values of the magnetization function can be recovered by the properties of the real system S_{N_0} without having to resort to the infinite system S_∞. In this way, the values obtained when taking the limit $N \longrightarrow \infty$ just yield approximations of the relevant property of the real target system. More importantly, as a theoretical analysis also shows for realistic values of N, the gradient in the derivatives of the free energy is sufficiently steep that the difference in the limit values of the thermodynamic quantities as $N \longrightarrow \infty$ and realistic systems with finite N_0 becomes negligibly small (Schmelzer and Ulbricht 1987; Fisher and Berker 1982).

A straightforward justification similar to the one obtained in cases of idealizations yielding approximations can now be given also for infinite limits that appear to give rise to "essential idealizations". In fact, the use of the latter is justified based on the two desirable properties of mathematical convenience and empirical adequacy. Here, though, we would like to emphasize a point that Butterfield does not develop, namely the fact that the choice of a certain topology over the other determines different degrees of empirical adequacy. For this purpose, let us consider the function f_n indexed by the parameter N that maps the independent variable x onto the real numbers: accordingly, the notation $f_n(x)$ indicates the value that the function takes on for each x in the domain, where in concrete physical cases the variable x would represent the possible states of the system under investigation. A natural topology is induced by the following type of convergence:

Pointwise Convergence: The sequence $\{f_n(x)\}_n$ converges pointwise to $f_\infty(x)$ if, given any x in $\mathbb{R}$ and given any $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon, x)$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $n > n_0(\varepsilon, x)$.

Translated into our framework, the fact that the sequence of functions $\{f_n(x)\}_n$ converges pointwise to the function $f_\infty(x)$ means that there is a real finite system S_{n_0} for which the value of the relevant function is approximated by the value of the limit function for a give state x, that is $f_{n_0}(x) \approx f_\infty(x)$. But one may as well adopt a different type of convergence, that is:

Uniform Convergence: The sequence $\{f_n(x)\}$ converges uniformly to $f_\infty(x)$ if, given any $\varepsilon > 0$, there exists a natural number $n_0(\varepsilon)$ such that, for any x in $\mathbb{R}$, $|f_n(x) - f(x)| < \varepsilon$ for every $n > n_0(\varepsilon)$.

Since in this case n_0 depends only on ε, and not even on the variable x like in the case of pointwise convergence, uniform convergence proves stronger than the latter (in fact, it is strictly stronger in that one can show by means of counter-examples that pointwise convergence does not imply uniform convergence). Indeed, here one can choose a natural number n_0 for which the sought-after approximation $f_{n_0}(x) \approx f_\infty(x)$ holds for all the possible states x. To the contrary, pointwise convergence allows for exceptions, in the sense that once n_0 is fixed together with the margin of approximation determined by $\varepsilon > 0$ there may be some state x for which $|f_n(x) - f(x)| < \varepsilon$ does not hold for any $n > n_0$. The upshot of this analysis is that the standard hierarchy of convergence conditions for functions representing physical quantities entails different degrees of accuracy up to which empirical accuracy is satisfied. Hence, just as the precise notion of approximation rests on what one means by the expression "sufficiently close", Butterfield's straightforward justification of infinite limits ultimately depends on the topology under which one takes the limits. In fact, in the explanation of first order phase transitions the criterion of empirical adequacy is satisfied only in its weak degrees of accuracy: for, it is just when one chooses the topology induced by pointwise convergence, and not by the stronger uniform convergence, that a sequence of continuous functions approaches a discontinuous function in the limit.

This analysis shows that the property of empirical adequacy is sensitive to the choice of topology. Thus, the extent to which one satisfies empirical correctness, namely the condition that the value of a function representing a given physical quantity is approximately equal to the empirical data, depends on how the limit is taken. One this point it is worth to making an important clarification: the issue whether the use of the limit is justified and the issue whether the use of the limit raises a threat to scientific realism, even though they have a common root, they should be kept separated. The common root is that the problem that, if the variable n represents a physical parameter in that, then when n becomes infinite the limit system would not be real. The former issue asks the question: why is one justified to use the infinite idealizations arising in the limit in physical applications? The second issue, instead, arises because it appears as if one violates statement (I) of the Paradox of Infinite Limits, which is a basic desideratum for Scientific Realism. Butterfield's two suggested properties, i.e. empirical adequacy and mathematical tractability, helps one answer the first question. Arguably, the use of an infinite limit is justified only if it is empirically adequate, that is if it yields empirically correct results, at least approximately. Furthermore, if one can recover empirically correct results also without taking the infinite limit but the latter is mathematically more tractable, then one has a pragmatic reason to favor the use of the unrealistic limit. That seems a *prima facie* reasonable justification for one to use an infinite idealization for practical calculations even though it does not, strictly speaking, represents the finite target system[12]. But, the fact that for pragmatic purposes one is justified to use the limit

[12]For a complete investigation of this issue one should actually discuss in much greater details than we can here the sense

in physical applications is independent from the issue whether one is a scientific realist or not. In fact, contrary to the issue of justification, this second issue bears on whether or not one ontologically commits to the existence of the limit system. When one wishes to give an explanation for some physical phenomenon P, if the only way to recover empirical correct results is by assuming the limit system, the resulting infinite idealization appears explanatorily indispensable, as prescribed by statement (II) in the Paradox of Infinite Limits. Scientific Realism is then called into question if one accepts the Enhanced Indispensability Argument, namely Statement (III) in the paradox, whereby the limit system is supposed to exist. Thus, it would seem that a scientific realist is not just justified to use the infinite limit, but also that she ought to be ontologically committed to it, in conflict with statement (I). Our strategy to resolve the problem, at least for the purported form of scientific realism that allows for approximate truth, is to reject statement (II) in the paradox, in that empirical correct results can be approximately obtained already on the way to the limit for the relevant physical quantities of interest.

5.2 Quantum Phase Transitions and Other Cases of Essential Idealizations

Let us move on to address the question whether the purported strategy to cope with the indispensability of infinite limits can be generalized to other cases where there appear essential idealizations. Butterfield (2011) conjectures that the solution of the Paradox of Phase Transitions proposed in the classical context holds generally. However, matters are less straightforward in other cases such as in the description of phase transitions in quantum statistical mechanics. In fact, in this case the indispensability of the thermodynamical limit seems to present itself in a stronger form than in classical statistical mechanics. A rigorous description of quantum phase transitions can be given within the algebraic approach to physical theories. In this framework, one represents a physical system by means of an algebra of observables, that is the set of physical quantities representing its observable properties. The possible states that the system can occupy are then defined on such an algebra. In some cases of interest, in order to describe the relevant physical quantities, instead of working directly with the algebra one needs to refer to its representation into a concrete Hilbert space (the so-called GNS representation), which is induced by a chosen state. For instance, in the Ising model, both the phase in which all spins in the chain are oriented in the up-direction and the phase in which all spins in the chain are oriented in the down-direction give rise to distinct concrete representations. Now, the familiar issue concerning the indispensability of infinite idealizations arises since a quantum observable representing magnetization can be rigorously defined only if one takes the thermodynamical limit. However, such an observable is state-dependent, in the sense that it is constructed only within the representation of one phase or, alternatively, within the representation of the other phase. This fact marks a difference with respect to the description of phase transitions in classical statistical mechanics. Indeed, in the classical context the values taken on by the magnetization function depend on the particular state of the system, which is determined by the applied magnetic field; instead, in the quantum context it is the magnetization function itself, and not just its values, that depends on a particular state.

Furthermore, in the case of quantum phase transitions, there is an additional problem arising since the representation induced by the state in which all spins are oriented in the up-direction and the representation induced by the state in which all spins are oriented in the down-direction are not unitarily equivalent. Without entering into too many technicalities, this means that, if one defines the magnetization observable with respect to one phase by taking the infinite limit $N \longrightarrow \infty$, one cannot recover the values of magnetization relative to the other phase (and viceversa), contrary to what happens for finite N's when the up and down representations always remain unitary equivalent. Arguably, the issue of unitarily inequivalence of the phase representations thus complicates the mystery of quantum phase transitions in the example of the Ising model. It goes beyond the scope of the present paper to settle this entire problem. For our purposes here, it is sufficient to point out that two contrasting positions can be

in which the limit system can be said to *represent* the target system. To this extent, Shech (2014) put forward a distinction between epistemologically and ontologically faithful representations. However, it goes beyond the scope of the present paper to survey the notion of representation.

identified in the literature: on the one hand, there are authors such as Liu and Emch (2005) and Ruetsche (2011), who claim that taking the thermodynamical limit to account for quantum phase transitions is indispensable; on the other hand, there are authors, most notably Landsman (2013) and Fraser (2016), who take a deflationary view towards the limit. In this respect, the jury is still out as to whether or not phase transitions in quantum statistical mechanics constitute a case of essential idealizations where one can provide an explanation of the phenomenon without committing to the reality of the infinite system constructed in the thermodynamical limit. [13]

Be that as it may, there is a further point that is worth to emphasizing regarding the general strategy to cope with other cases of essential idealizations, beside classical phase transitions. As pointed out by Palacios (2018), it does not suffice to demonstrate that approximately the same behavior that occurs in the limit, also occurs "on the way to the limit", but in addition we need to demonstrate that it arises for realistic values of the parameter n that goes to infinity. This latter condition is important because there are cases, such as the ergodic approach to equilibrium, in which the expected behavior arises for finite but unrealistic values of the parameter that goes to infinity. In more detail, in the ergodic theory of equilibrium, one takes the infinite-time limit $t \longrightarrow \infty$ in order to assure that phase-averages and time-averages coincide. Even in simple examples like a small sample of diluted hydrogen, though, one can estimate that the desired behavior can occur for times t_0 that are finite but unimaginably longer than the age of universe. In such cases, even if we can actually demonstrate that the behavior arises for finite values of the parameter, we would not have succeeded in demonstrating the empirical adequacy of the theory.

To conclude, the analysis we have developed in the present section indicates that the challenge posed by essential idealizations to scientific realism ought to be resolved on a case-by-case basis and that ultimately calls for empirical considerations. In general, the strategy to dispense from the infinite-limit system in the explanation of a certain phenomenon involves a suitable selection of the physical quantities to focus on, so as to yield approximations of the relevant properties of the target system already "on the way to the limit", where the degrees of inexactness that one may accept depends on the particular situation at stake and the choice of an adequate topology. Accordingly, as the example of classical phase transitions shows, one can assuage worries concerning scientific realism. In fact, if the idealization is not genuinely essential, one does not need to assume statement (II) of the Paradox of Infinite Limits. Moreover, the limit values of the relevant physical quantities can still give good, albeit inexact, descriptions of the properties of realistic systems, which are supposed to be finite in accordance with statement (I), and so in order to evade the paradox we do not even need to cast doubts onto the validity of the enhanced indispensability argument contained in statement (III). The Paradox of Infinite Limits can thus be avoided in spite of apparent essential idealizations emerging in the infinite limit, while endorsing a form of scientific realism that allows for our best theories to be sufficiently close to the truth. There now remains to discuss the last type of use of infinite limits, namely as abstractions. As we will see, differently from the case-studies we have investigated so far, sometimes in such cases the variable n growing to infinity does not represent a physical parameter, and hence in principle one may not satisfy statement (I) in the paradox.

6 Infinite Limits as Abstractions

For the sake of completeness, in this last section we address a different role played by mathematical limits that has been much less discussed in the philosophical literature than approximations and idealizations, namely the use of limits as *abstractions*. We mentioned above that the thermodynamic limit, in which the number of particles N as well as the volume V go to infinity, can be used to recover the quantities that successfully describe phase transitions in thermodynamics. Another important role of the thermodynamic limit is to enable the removal of irrelevant contributions such as 'surface' and 'edges' effects. More to the point, any finite lattice system will include contributions to the partition function

[13] Another example of apparent essential idealizations are continuous phase transitions. We will address this case in Section 6.

coming from the edges and surfaces, which may be considerably different from those coming from the center of the sample. Taking the thermodynamic limit allows one to treat the system as a bulk, leaving out surface effects that are mostly irrelevant for the behavior. In this sense, the thermodynamic limit enables one to abstract away details that do not make a difference for the phenomenon under investigation (See also (Mainwood 2006; Butterfield 2011; Jones 2006)).

A more interesting case of infinite limits used as abstractions is given in the context of continuous phase transitions. In contrast to first-order phase transitions that involve discontinuities in the derivatives of the free energy, in the case of continuous phase transitions there are no discontinuities but rather there are divergences in the response functions (e.g. specific heat, susceptibility for a magnet, compressibility for a fluid). An example of a continuous phase transition is the transition in magnetic materials from the phase featuring spontaneous magnetization – the ferromagnetic phase – to the phase where the spontaneous magnetization vanishes – the paramagnetic phase. Continuous phase transitions are also characterized by the divergence of a quantity called the correlation length ξ, which measures the distance over which the particles are correlated. A typical way of dealing with these long correlations is by introducing renormalization group methods, which are mathematical and conceptual tools that consist in defining a transformation that successively coarse-grains the effective degrees of freedom while keeping the partition function and the free energy (approximately) invariant. Palacios (2019) pointed out that such a process can be interpreted as assuming an infinite limit, in which the number of iterations n of the renormalization group transformation goes to infinity. An important aspect of the infinite limit for $n \longrightarrow \infty$ is that, at each step of the sequence, irrelevant coupling constants are abstracted away, so as to retain only factors that are relevant for the behavior under description. That follows because, in this process, the partition function and the free energy that determine the behavior at a phase transition remain (approximately) invariant. After an infinite number of iterations, the sequence of systems may converge towards non-trivial fixed points, where the values of the coupling constants no longer change by applying the transformation. Note that the limit for the number n of iterations going to infinite does not represent any physical parameter of the system under description, but rather the number of transformations that one needs to apply by means of the renormalization group procedure.

The beauty of renormalization group methods is that linearizing around fixed points allows one to calculate the critical exponents of the power laws that determine the behavior close to the transition. Furthermore, it allows one to give an account of universality, namely the remarkable fact that physical systems as heterogeneous as fluids and magnets exhibit the same behavior near a phase transition (details in (Goldenfeld 1992)). There is good reason to consider the infinite iteration limit $n \longrightarrow \infty$ as an abstraction. In fact, in section 2, we defined abstraction as a process that consists in leaving some factors out without misrepresenting the properties of the original system. In fact, when applying a renormalization group transformation to the description of a given system, one does exactly this: that is, at each stage n, one leaves out irrelevant details while retaining those factors that make a difference for the phenomenon to be explained. To be sure, in doing that, one might have to use approximations; yet, one does not introduce any idealization. Moreover, it is also important to point out that the renormalization group process has the same epistemic role attributed to abstractions, namely that it enables us to give an explanation of the common behavior generated by systems that are diverse from each other at a fine grained level, thereby accounting for universality.

In recent philosophical literature, the use of renormalization group methods has often been considered as an example of "essential idealizations" (e.g. Batterman (2011, 2017) and (Morrison 2012). Indeed, Batterman (2017, p. 571) says in a footnote:

> It seems to me that if one is going to hold that the use of the in finite limits is a convenience, then one should be able to say how (even if inconveniently) one might go about finding a fixed point of the RG transformation without infinite iterations. I have not seen any sketch of how this is to be done. The point is that the fixed point, as just noted, determines the behavior of the flow in its neighborhood. If we want to explain the universal behavior of finite but large systems using the RG, then we need to find a fixed point and, to my knowledge, this requires an infinite system.

And, later on in the paper (p. 573), he argues

> Kadanoff's understanding of the new RG theory of critical phenomena reflects a different conception of the role of asymptotics and infinities. The kind of upscaling that leads to an understanding of the universal macroscopic behavior of micro-diverse systems is different than the upscaling provided in the ensemble averaging of mean field theory.

However, we resist the conclusion that the appeal to renormalization group theory gives rise to an essential idealization. For the purpose of our argument, it is useful to distinguish between the roles played by the different limits involved in this approach, namely the thermodynamical limit entailing $N \longrightarrow \infty$ for the number of molecules and the limit $n \longrightarrow \infty$ for the number of iterations employed in the use of renormalization group theory. As it is well known, in order to define a system with an infinite correlation length, one needs to take the thermodynamic limit. Granted, the latter may perhaps be interpreted as an idealization, in the sense that it gives rise to a fictional system with an infinite number of particles. Nonetheless, as Palacios (2019) pointed out, this limit, which appears essential to find the non-trivial fixed points that explain critical phenomena, can be also demoted to an approximation. In order to understand why, one should note that the two limits involved in the explanation of critical phenomena, i.e. $N \longrightarrow \infty$ and $n \longrightarrow \infty$, do not commute with each other: indeed, taking the limit $N \longrightarrow \infty$ followed by the limit $n \to \infty$ yields empirically correct results, whereas taking the limit $n \longrightarrow \infty$ before $N \longrightarrow \infty$ yields results that are not empirically correct. What Palacios emphasized is that for finite systems, i.e. when $N < \infty$, one should *not* take the second infinite limit if one wants to obtain empirically correct results. In fact, she showed – based on the work done by Wilson and Kogut (1974) – that in finite systems one can find effective fixed point solutions that approximate the desired behavior for a large but finite number n of iterations, which means that one obtains empirically correct results "on the way" to the second limit $n \longrightarrow \infty$ as well as "on the way" to the thermodynamic limit $N \longrightarrow \infty$.

These results are relevant for our answer to above question because they indicate that the Paradox of Infinite Limits can again be resolved by resorting to the notion of approximations, like in the examples discussed in the previous sections. Note, first of all, that the variable n corresponding to the number of iterations is not a physical parameter, hence in principle one may not violate statement (I) even if one takes the limit $n \longrightarrow \infty$. However, since this limit does not commute with the thermodynamical limit, taking the former implies taking the latter limit too, and thus if n grows to infinity so does the number N of particles in the gas, thereby violating statement (I). Instead, the resolution to the paradox comes once again by rejecting statement (II), which says that the explanation of the phenomenon P can *only* be given by means of claims about an infinite system constructed in the limit $n \longrightarrow \infty$. In fact, the procedure outlined above shows how the two limits $N \longrightarrow \infty$ and $n \longrightarrow \infty$ can be dispensed by means of the notion of approximation: accordingly, one can satisfy the condition of empirically correctness without committing to the limit system S_∞. In light of this analysis, we conclude that one has good reason to believe that renormalization group methods, at least in the example we considered, do not pose by themselves any challenge for a form of scientific realism allowing for approximate truth.

7 Conclusion

In this paper, we addressed the issue of whether or not the use of infinite limits in physics raises a problem of compatibility with scientific realism. For this purpose, we surveyed various physical examples where infinite limits are invoked in order to describe real target systems, so as to offer a taxonomy of the different uses of infinite limits (section 3). In particular, we distinguished between approximations that do not constitute idealizations, idealizations yielding approximations, essential idealizations and abstractions, which we then discussed in greater details in the subsequent sections. We argued in section 5 that a challenge for scientific realism arises just when infinite limits are intended as essential idealizations, due to the fact that it appears as if empirically correct results can be recovered only in the limit. However, if one commits to the limit system in that it seems indispensable, one runs against the

Paradox of Infinite Limits since the limit system is infinite whereas real systems are necessarily finite. We then went on to suggest how the ensuing worries for scientific realism can be assuaged in concrete examples, e.g. in the much debated case of classical phase transitions. The strategy to do so is to show, without referring to the infinite system, that the limit values of some physical quantities of interest yield approximations of the values obtained for realistic systems. That is of course compatible with a form of scientific realism that allows for theories to be just sufficiently close to the truth. In the same vein, as we explained in section 4 and section 6 by means of physical examples such as that of thermodynamically reversible processes and that of continuous phase transitions, understanding infinite limits as approximations that do not constitute idealizations or even as abstractions, respectively, does not raise any further issue for scientific realism.

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